On Classical Set-Compatibility

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Abstract

In this paper, I generalise the logical concept of compatibility into a broader set-theoretical one. The basic idea is that two sets are incompatible if they *produce* at least one pair of opposite objects under some operation. I formalise opposition as an operation $' : \mathbf{E} \longrightarrow \mathbf{E}$, where $\mathbf{E}$ is the set of opposable elements of our universe $\mathbf{U}$, and I propose some models. From this, I define a relation $\mathsf{C} : \wp\mathbf{U} \times \wp\mathbf{U} \times \wp\mathbf{U}^{\wp\mathbf{U}}$, which has (mutual) logical compatibility as its more natural interpretation.

Keywords: consistency, paraconsistency, opposition, involution.

3.1. Opposition

That some set is compatible with another depends on whether they *produce* a pair of *opposite* objects. If we assume that any x is the opposite of its own opposite, an operation of opposition $' : \mathbf{E} \longrightarrow \mathbf{E}$ has to be involutory:

Axiom 3.1. $x = x''$,

from where we obtain the following corollaries:

Corollary 3.2. If $x' = y'$, then $x = y$.

Corollary 3.3. If $x' = y$, then $x = y'$.

The first means that $'$ is *injective* and, since $'$ is its own inverse function, we can interpret 3.3 as saying that $'$ is *surjective*, which establishes that $'$ is *bijective*.

Intuitively, x' denotes the opposite of x, i.e., an operation of opposition transforms an element of $\mathbf{E}$ into its opposite. Since it is not necessary that all elements

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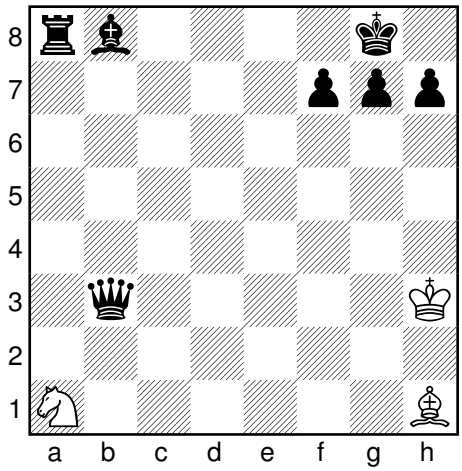

Figure 3.1: The chess table.

of our universe **U** have an opposite, the domain of this operation is restricted to **E**, which is accordingly interpreted as the set of *opposable* elements of **U**.

These properties, however, are not sufficient for completely characterising the concept of opposition. Some additional properties depend on the introduction of other operations. In fact, there is room for debating whether some of these properties are adequate or not. For example, we might say that the white bishop in h1 (see figure 3.1) is opposed to the black rook in a8, but not the other way around. Following the same intuition, we can also state that this rook is opposed to the white knight in a1 (which is opposed to the black queen in b3?).

This understanding of the concept of opposition may be fairly considered mistaken, but it shows to some extent that the properties of our formalisation depend on what notion of opposition are we trying to capture. However, I do not pretend to capture all possible senses of opposition in this short paper. In fact, I will add a further restriction to this concept by requiring that $'$ be irreflexive, preventing every x from being its own opposite:

Postulate 3.4. $x \neq x'$.

3.2. Interpretations of $'$

Several mathematical functions are relations of opposition. For example, *classical negation* satisfies all the previous properties. Furthermore, if we take our domain **U** to be the set of wffs of a language, we have that **E** = **U**. If p and q are wffs, we may take $p = q$ to mean that p and q have the same logical value.

The *inverse operation* of group theory is also an operation of opposition. This implies that the *additive inverse* ($x' \mapsto -x$) and the *multiplicative inverse*

$(x' \mapsto 1/x)$ are operations of opposition in the relevant domains. Clearly, this also holds for the inverse operation of any model of group theory.

A remarkable interpretation is the *absolute complement operation* $(\mathbf{A}' \mapsto \mathbf{A}^C)$ of set theory, which means that we can also have *properties* in the domain of $'$. For example, if P is in the domain of $'$, we may define P' as the predicate in whose extension are all x such that $\neg \mathrm{P}x$; that is, the extension of P' is the absolute complement of the set corresponding to the extension of P.

We can alternatively interpret P' as the one antonymous property of P, which is the one antonymous property of P'. The set of opposable elements would then be constituted only by properties that could appear in a pair of reverse antonyms such as good/bad, beautiful/ugly, dark/bright, and the like.

In the first approach, if P stands for "x is transparent", P' would stand for "x is opaque", since all non transparent things are opaque (assuming our universe consists of normal-sized physical objects). In the second approach, if P stands for "x is dark", P' could stand for "x is bright".

3.3. The Relation C

Classical compatibility can be defined from opposition as a three-place relation $\mathsf{C} : \wp\mathbf{U} \times \wp\mathbf{U} \times \wp\mathbf{U}^{\wp\mathbf{U}}$ satisfying:[38]

Definition 3.5. $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$ iff $x, x' \in (\mathbf{A} \cup \mathbf{B})^*$, for no x.

The expression $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$ is read "$\mathbf{A}$ is *compatible* with $\mathbf{B}$ with respect to $*$," or "$\mathbf{A}$ is $*$-*compatible* with $\mathbf{B}$," where $\mathbf{A}, \mathbf{B} \in \wp\mathbf{U}$ and $* \in \wp\mathbf{U}^{\wp\mathbf{U}}$. Accordingly, we say that $\mathbf{A}$ is $*$-*incompatible* with $\mathbf{B}$ iff not $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$. The relation C is *symmetric* in the sense that:

Theorem 3.6. $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$ iff $\mathsf{C}(\mathbf{B}, \mathbf{A})^*$, for all $\mathbf{A}, \mathbf{B}$ and $*$.

However, C is no equivalence relation since it is neither *reflexive* nor *transitive*. For example, if $*$ was a closure operation, it could not be reflexive since $\mathsf{C}(\{x, x'\}, \{x, x'\})^*$ never holds. Nor can it be transitive because even when both $\mathsf{C}(\{a\}, \{b\})^*$ and $\mathsf{C}(\{b\}, \{a'\})^*$ hold, $\mathsf{C}(\{a\}, \{a'\})^*$ does not.

3.4. Interpretations of C

The most natural interpretation of C is given in the context of logic and is related to the concept of consistency. We often say that a set of sentences is consistent iff it does not imply any pair of mutually contradictory statements.

Definition 3.7. $\mathbf{A}$ is *consistent* with respect to (the consequence relation) $\vdash$ iff there is no α for which both $\mathbf{A} \vdash \alpha$ and $\mathbf{A} \vdash \neg\alpha$, and *inconsistent* otherwise.

[38] Remember that Y^X is the set of all functions from X to Y. Hence, $\wp\mathbf{U}^{\wp\mathbf{U}}$ is the set of all functions from and to sets of $\mathbf{U}$.

Considering that $\mathbf{A}^{\vdash} = \{\alpha \mid \mathbf{A} \vdash \alpha\}$, this definition is equivalent to:

Definition 3.8. $\mathbf{A}$ is *consistent* with respect to $\vdash$ iff $\alpha, \neg\alpha \in \mathbf{A}^{\vdash}$ holds for no α, and *inconsistent* otherwise.

This put us one step away from our logical interpretation of C, which can be done from the concept of mutual consistency.

Definition 3.9. $\mathbf{A}$ and $\mathbf{B}$ are *mutually consistent* with respect to $\vdash$ iff $\alpha, \neg\alpha \in (\mathbf{A} \cup \mathbf{B})^{\vdash}$ holds for no α, and *mutually inconsistent* otherwise.

Now, for our interpretation we take $\mathbf{U}$ to be the set of statements or propositions of a formal language, the function $'$, the operation of logical negation, and $*$, a relation of logical consequence ($\vdash: \wp\mathbf{U} \longrightarrow \mathbf{U}$). From this, it follows that two sets of sentences $\mathbf{A}$ and $\mathbf{B}$ are compatible iff there is no α such that $\mathbf{A} \cup \mathbf{B} \vdash \alpha$ and $\mathbf{A} \cup \mathbf{B} \vdash \neg\alpha$. That is, in order for two sets of sentences to be consistent, it is necessary that the set of their logical consequences be consistent too.

Let us compare this definition with that of Batens and Meheus (2000). Although they initially define compatibility as a relation between sentences and sets of sentences, they clarify in their footnote 1 that it is a symmetric relation. The following definition sufficiently captures their syntactic definition of compatibility.

Definition 3.10. $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$ iff, for all α, $\mathbf{A} \vdash \alpha$ implies $\mathbf{B} \nvdash \neg\alpha$.

If $\vdash$ is monotone, then any pair of sets in the extension of C is also in the extension of D.

Theorem 3.11. If $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$, then $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$.

Proof. Assume $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$ and let $\mathbf{A} \vdash \alpha$. Since $\vdash$ is monotone, it follows that $\mathbf{A} \cup \mathbf{B} \vdash \alpha$. By the same property, it would follow from $\mathbf{B} \vdash \neg\alpha$ that $\mathbf{A} \cup \mathbf{B} \vdash \neg\alpha$, which is forbidden by $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$. Hence, $\mathbf{B} \nvdash \neg\alpha$. □

The converse follows if $\vdash$ is a classical relation, in which case it follows that it is monotone and satisfies the *compactness theorem*.

Theorem 3.12. If $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$, then $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$.

Proof. We assume $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$ and suppose for *reductio* that $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ is inconsistent. In that case, the compactness theorem guarantees that $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ has an inconsistent subset. Since $\vdash$ is classical, $\mathbf{A}^{\vdash}$ is consistent, otherwise $\mathbf{A}$ would imply all formulae, including the negations of tautologies, and since $\mathbf{B}$ implies all tautologies, this would mean that not $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$. We can prove that $\mathbf{B}$ is consistent in a similar way. Hence, in order for $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ to be inconsistent it is necessary that some α be such that $\mathbf{A} \vdash \alpha$ and $\mathbf{B} \vdash \neg\alpha$, which is forbidden by $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$. □

This proves that (classical) logical compatibility, as defined by Batens and Meheus, is a model of C. Let us now turn to other interpretations.

In another interpretation, we may speak about incompatible sets of entities when $\mathbf{U}$ is interpreted as the set of all (conceivable) entities. One way to show this is through answering the question triggering the *irresistible force paradox*, i.e. *what happens when an unstoppable force meets an immovable object*? Let M stand for the relation "x can move y". An immovable object can be then characterised as any y satisfying $\forall x(\neg x\mathrm{M}y)$. An unstoppable force is instead an object that can move any object that encounters; that is, an x satisfying $\forall y(x\mathrm{M}y)$.

Now, is it possible that an unstoppable force and an immovable object thus defined can exist in the same possible world? Unless we dismiss the principle of non contradiction, the answer is clearly no. Otherwise, if there were an object, say a, such that $\forall y(a\mathrm{M}y)$, and another b such that $\forall x(\neg x\mathrm{M}b)$, it would follow that both $a\mathrm{M}b$ and $\neg a\mathrm{M}b$. In this sense, we can say that the opposite of an immovable object is an unstoppable force, which makes mutually incompatible any two sets that can produce both.[39]

For our last interpretation, it is possible to state that two sets of properties are compatible or incompatible for a given entity. In order to do this we can treat entities as sets of properties: the properties that those entities have. This treatment corresponds to Russell's conception of proper names, for whom "what would commonly be called a 'thing' is nothing but a *bundle of coexisting qualities* such as redness, hardness, etc." (1995, p. 97, my emphasis). For example, if we let B stand for "x is single" and M for "x is married", we may say that $\mathrm{B}' = \mathrm{M}$. The properties of being single and being married are in this sense incompatible, since all non married persons are single.

3.5. Limitations of Classical Set-Compatibility

It may be argued that against this proposal that C fails to be reflexive, when it should be so. After all, how can a set be incompatible with itself? Let us notice, though, that $\mathsf{C}(\mathbf{A}, \mathbf{A})^*$ only fails for those $\mathbf{A}$ such that $x, x' \in \mathbf{A}^*$, for some x.

Corollary 3.13. $\mathsf{C}(\mathbf{A}, \mathbf{A})^*$ iff $x, x' \in \mathbf{A}^*$, for no x.

In this framework we can state that all sets that are incompatible with themselves are unacceptable or inconceivable, depending on the kind of incompatibility we are talking about. What is more, the existence of self-incompatible sets would be a feature of this proposal in that it would be a formal way to characterise such unacceptable and inconceivable sets.

A more important limitation of this approach is that it would make it impossible to analyse (in)compatibility between inconsistent sets. For example, if we had

[39]The core of this solution was proposed by Isaac Asimov in his *Book of Facts*: "The rules of the game of reason say the question is meaningless and requires no answer. The question: 'What would happen if an irresistible force met an immovable body?' In a universe where one of the above conditions exists, by definition the other cannot exist" (1979, p. 281).

an inconsistent (though non trivial) theory **T**, we would have to conclude that all sets of observation statements (consistent or otherwise) are incompatible with **T**. This would result in **T** being a priori false instead of falsifiable, which does not need to be the case as I show in Bartolo Alegre (2019).

As it happens, this situation can be corrected for if we stick Batens' and Meheus' definition. In such case, though, compatibility could not be a symmetric relation, as they want it. One such theory of *para-compatibility* is a topic for another paper.

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